Solving for the dynamics of the universe

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Abstract

A new method of solving the Einstein–Friedmann dynamical equations of a spatially homogeneous and isotropic universe is presented. The method is applicable when the equation of state of the material content assumes the form $P = (\gamma - 1)\,\rho$, $\gamma =$ constant. The solution for the expansion factor is commonly given only for $\gamma = 0, 1$ and $4/3$ when the curvature index is $K = \pm 1$. The proposed procedure is valid for *general* γ and K and it allows for ease of derivation of the solutions. This alternative method is useful pedagogically to introduce basic cosmology.

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1 Introduction

The Friedmann–Lemaitre–Robertson–Walker (FLRW) solutions to the Einstein field equations of general relativity are a cornerstone in the development of modern cosmology. The FLRW metric describes a spatially homogeneous and isotropic universe satisfying the Copernican principle, and is the starting point for studying dynamical models of the universe. To this end, one must specify the nature of the matter which is the source of gravitation by assigning the corresponding equation of state, which varies during different epochs of the history of the universe.

The FLRW metric is given, in comoving coordinates (t, r, θ, φ), by the line element

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + f^2(r) \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right] \ , \tag{1.1}$$

where the three possible elementary topologies are classified according to a normalized curvature index K which assumes the values $0, \pm 1$, and

$$f(r) = \sinh r \qquad\qquad (K = -1) \tag{1.2}$$

for the open universe,

$$f(r) = r \qquad\qquad (K = 0) \tag{1.3}$$

for the critical universe, and

$$f(r) = \sin r \qquad\qquad (K = +1) \tag{1.4}$$

for the closed universe. Other topologies are possible (see e.g. Ref. 1, p. 725). The function $a(t)$ of the comoving time t (the "scale factor") is determined by the Einstein–Friedmann dynamical equations[2–7]

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P) \ , \tag{1.5}$$

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G\rho}{3} - \frac{K}{a^2} \ , \tag{1.6}$$

where ρ and P are, respectively, the energy density and pressure of the material content of the universe, which is assumed to be a perfect fluid. An overdot denotes differentiation with respect to the comoving time t, G is Newton's constant and units are used in which the speed of light in vacuum assumes the value unity. It is further assumed that the cosmological constant vanishes.

One can solve the Einstein–Friedmann equations (1.5) and (1.6) once a barotropic equation of state $P = P(\rho)$ is given. In many situations of physical interest, the equation of state assumes the form

$$P = (\gamma - 1)\, \rho \,, \qquad \gamma = \text{constant} \,. \tag{1.7}$$

This assumption reproduces important epochs in the history of the universe. For $\gamma = 1$ one obtains the "dust" equation of state $P = 0$ of the matter–dominated epoch; for $\gamma = 4/3$, the radiation equation of state $P = \rho/3$ of the radiation–dominated era; for $\gamma = 0$, the vacuum equation of state $P = -\rho$ of inflation. For $\gamma = 2/3$, the curvature–dominated coasting universe with $a(t) = a_0 t$ is obtained.

When the universe is spatially flat ($K = 0$), the integration of the dynamical equations (1.5), (1.6) with the assumption (1.7) is straightforward[2] for any value of the constant γ:

$$a(t) = a_0 t^{\frac{2}{3\gamma}} \qquad (\gamma \neq 0) \,, \tag{1.8}$$

$$a(t) = a_0 \mathrm{e}^{Ht} \,, \qquad \dot{H} = 0 \qquad (\gamma = 0) \,. \tag{1.9}$$

However, for $K = \pm 1$, the solution is given (explicitly or in parametric form) only for the special values $\gamma = 1, 4/3$ in old and recent textbooks[1−7].

It is actually not difficult to derive a solution for any nonzero value of the constant γ using a standard procedure[3] which is general, i.e. it does not depend upon the assumption $P = (\gamma - 1)\, \rho$. Section 2 presents a summary of the usual method for deriving the scale factor $a(t)$. An alternative method, which consists in reducing the Einstein–Friedmann equations to a Riccati equation, and is applicable when Eq. (1.7) holds, is explained in Sec. 3. From the mathematical point of view, the latter method avoids the consideration of the energy conservation equation

$$\dot{\rho} + 3\,(P + \rho)\, \frac{\dot{a}}{a} = 0 \tag{1.10}$$

and the calculation of an indefinite integral; rather, one solves a nonlinear Riccati equation. The alternative approach is more direct than the standard one and the gain in clarity of exposition makes it more suitable for an introductory cosmology course. Section 4 presents a brief discussion of the equation of state $P = (\gamma - 1)\, \rho$ and the conclusions.

2 The standard derivation of the scale factor

The standard method to obtain the scale factor a of the universe proceeds as follows[3]. The energy conservation equation (1.10) yields

$$3\ln a = -\int \frac{d\rho}{P+\rho} + \text{constant} \tag{2.1}$$

for $\gamma \neq 0$. Upon use of the conformal time η defined by

$$dt = a(\eta) d\eta \; , \tag{2.2}$$

the Einstein–Friedmann equations (1.5), (1.6) yield

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{8\pi G}{3}\rho a^2 - K}} \; . \tag{2.3}$$

In order to obtain $a(\eta)$, one prescribes the equation of state $P = P(\rho)$, solves Eq. (2.1) and inverts it obtaining $\rho = \rho(a)$. Further substitution into Eq. (2.3) and inversion provide $a = a(\eta)$. Integration of Eq. (2.2) provides the comoving time $t(\eta)$ as a function of conformal time. The scale factor is then expressed in parametric form $(a(\eta), t(\eta))$. Sometimes it is possible to eliminate the parametric dependence on η and obtain the expansion factor as an explicit function $a(t)$ of comoving time.

The method is quite general; as a particular case, it can be applied when the equation of state is of the form $P = (\gamma - 1)\,\rho$ with constant nonvanishing γ. Equations (2.1) and (2.3) then yield

$$a^3 \rho^{1/\gamma} = \text{constant} \; , \tag{2.4}$$

$$\eta = \pm \int \frac{da}{a\sqrt{\frac{8\pi G}{3} C_1 a^{2-3\gamma} - K}} \tag{2.5}$$

for $\gamma \neq 0, 2/3$, where C_1 is an integration constant. By introducing the variable

$$x = \left(\frac{8\pi G C_1}{3} \right)^{\frac{1}{2-3\gamma}} a \; , \tag{2.6}$$

and using

$$\int \frac{dx}{x\sqrt{x^n+1}} = \frac{1}{n} \ln \left(\frac{\sqrt{x^n+1}-1}{\sqrt{x^n+1}+1} \right) \; , \tag{2.7}$$

$$\int \frac{dx}{x\sqrt{x^n - 1}} = \frac{2}{n} \operatorname{arcsec}\left(x^{n/2}\right) \ , \tag{2.8}$$

one integrates and inverts Eq. (2.5) to obtain

$$a(\eta) = a_0 \sinh^{1/c}(c\eta) \ , \tag{2.9}$$

$$t(\eta) = a_0 \int_0^\eta d\eta' \sinh^{1/c}(c\eta') \ , \tag{2.10}$$

for $K = -1$. a_0 and c are constants, with

$$c = \frac{3}{2}\gamma - 1 \ , \tag{2.11}$$

and the boundary condition $a\,(\eta = 0) = 0$ has been imposed.

Similarly, for $K = +1$, one obtains

$$a(\eta) = a_0 \left[\cos\left(c\eta + d\right)\right]^{1/c} \ , \tag{2.12}$$

$$t(\eta) = a_0 \int_0^\eta d\eta' \left[\cos\left(c\eta' + d\right)\right]^{1/c} \ . \tag{2.13}$$

For $\gamma = 2/3$ and $K = -1$ one obtains a curvature–dominated universe for which Eq. (1.6) is approximated by $(\dot{a}/a)^2 \simeq -K/a^2$. In this case, Eq. (2.3) yields $a = a_0 \exp(\beta\eta)$ ($\beta =$ constant), and $t = t_0 e^{\eta\beta}$ gives $a = a_0 t$. It is easier to obtain this form of the scale factor directly from Eq. (1.5), which reduces to $\ddot{a} = 0$.

The solutions (2.9), (2.10) and (2.12), (2.13) for the scale factor are presented in the textbooks only for the special values 1 and 4/3 of the constant γ. For $\gamma = 4/3$ one eliminates the parameter η to obtain

$$a(t) = a_0 \left[1 - \left(1 - \frac{t}{t_0}\right)^2\right]^{1/2} \tag{2.14}$$

for $K = +1$, and

$$a(t) = a_0 \left[\left(1 + \frac{t}{t_0}\right)^2 - 1\right]^{1/2} \tag{2.15}$$

for $K = -1$.

The standard solution method of the Einstein–Friedmann equations has the virtue of being general; it does not rely upon the assumption (1.7). However, the need to invert Eqs. (2.1) and (2.3) and to compute the indefinite integrals (2.6), (2.7) detracts from the elegance and clarity that is possible when the ratio P/ρ is constant. The latter condition is satisfied in many physically important situations.

3 An alternative method

There is an alternative procedure to derive the scale factor for a general value of γ when the equation of state of the universe's material content is given by $P = (\gamma - 1)\,\rho$ with $\gamma =$ constant, which covers many cases of physical interest. Being straightforward, this new method is valuable for pedagogical purposes and proceeds as follows: Eqs. (1.5), (1.6) and (1.7) yield

$$\frac{\ddot{a}}{a} + c\left(\frac{\dot{a}}{a}\right)^2 + \frac{cK}{a^2} = 0 \; , \tag{3.1}$$

with c given by Eq. (2.11). For $K = 0$, Eq. (3.1) is immediately integrated to give Eqs. (1.8), (1.9). For $K = \pm 1$, Eq. (3.1) is rewritten as

$$\frac{a''}{a} + (c-1)\left(\frac{a'}{a}\right)^2 + cK = 0 \; , \tag{3.2}$$

by making use of the conformal time η, and where a prime denotes differentiation with respect to η. By employing the variable

$$u \equiv \frac{a'}{a} \; , \tag{3.3}$$

Eq. (3.2) becomes

$$u' + cu^2 + Kc = 0 \; , \tag{3.4}$$

which is a Riccati equation. The Riccati equation, which has the general form

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x) \tag{3.5}$$

where $y = y(x)$, has been the subject of many studies in the theory of ordinary differential equations, and can be solved explicitly[8,9]. The solution is found by introducing the variable w defined by

$$u = \frac{1}{c}\,\frac{w'}{w} \; , \tag{3.6}$$

which changes Eq. (3.4) to

$$w'' + Kc^2 w = 0 \; , \tag{3.7}$$

the solution of which is trivial. For $K = +1$, one finds the solutions (2.12), (2.13), while for $K = -1$, one recovers (2.9), (2.10).

The alternative method is applicable when the equation of state assumes the form $P = (\gamma - 1)\,\rho$ and the cosmological constant vanishes. With these conditions satisfied, the alternative solution procedure is more direct than the general method. It is now appropriate to comment on the equation of state $P = (\gamma - 1)\,\rho$, $\gamma =$ constant, which reproduces several situations of significant physical interest.

4 Discussion

The assumption that the equation of state is of the form $P = (\gamma - 1)\,\rho$ with constant γ is justified in many important situations which describe the standard big–bang cosmology. However, it is important to realize that Eq. (1.7) is a strong assumption and by no means yields the most general solution for the scale factor of a FLRW universe. To make a physically interesting example, consider the inflationary (i.e. $\ddot{a} > 0$) epoch of the early universe in the $K = 0$ case. Many inflationary scenarios are known[10], corresponding to different concave shapes of the scale factor $a(t)$; the assumption (1.7) allows the solutions

$$a(t) = a_0 t^{\frac{2}{3\gamma}} \tag{4.1}$$

("power–law inflation") for $0 < \gamma < 2/3$, and

$$a = a_0 \mathrm{e}^{Ht} \,, \qquad \dot{H} = 0 \tag{4.2}$$

for $\gamma = 0$. Vice–versa, using the dynamical equations (1.5), (1.6), it is straightforward to prove that the latter solutions imply $P/\rho =$ constant. The assumption (1.7) reproduces *only* exponential expansion (the prototype of inflation) and power–law inflation. All the other inflationary scenarios correspond to a $\gamma(t)$ which changes with time during inflation. A time–dependent $\gamma(t)$ can also be used to describe a non–interacting mixture of dust and radiation; however the method of solution of the Einstein–Friedmann equations (1.5), (1.6) presented in Sec. 3 applies only when γ is constant, and when the cosmological constant vanishes. If the cosmological constant is nonzero, Eq. (3.1) does not reduce to a Riccati equation.

The limitations of the assumption $\gamma =$ constant are thus made clear: while this new approach to the Einstein–Friedmann equations does not replace the standard approach, it is more direct and is preferable in an introduction to cosmology. Its value lies in the ease of demonstration of the solutions, which is crucial for students to grasp the basic concepts of cosmology.

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